IMM-NYU 305
NOVEMBER 1962

NEW YORK UNIVERSITY
COURANT INSTITUTE
MATHEMATICAL SCIENCES

On the Growth of Solutions of Quasi-linear Parabolic Equations

STANLEY KAPLAN

PREPARED UNDER GRANT NSF-G14520
NATIONAL SCIENCE FOUNDATION

IMM-NYU 305
November 1962

New York University

Courant Institute of Mathematical Sciences

ON THE GROWTH OF SOLUTIONS OF QUASI-LINEAR PARABOLIC EQUATIONS

STANLEY KAPLAN

This report represents results obtained under the sponsorship of the National Science Foundation, Grant No. NSF-G 14520.

Introduction

We deal here with various properties of the solutions of a large class of mixed initial-boundary value problems for the parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)\,\frac{\partial^2 u}{\partial x_i \partial x_j} = F(x,t,u,\nabla u). \qquad (1)$$

In particular, we discuss some aspects of the theory of uniqueness, stability, and behavior in the large for solutions of (1), trying, wherever possible, to reduce our problem by means of an elementary device to the study of an ordinary differential equation of the form

$$\frac{du}{dt} = G(t,u). \qquad (2)$$

In fact, we might alternately have entitled this work "What one may say about quasi-linear parabolic equations without using anything deep like Greens' functions, Schauder estimates, or Nash's theorem".

First, in Section 1, we prove a comparison theorem for (1) in cylindrical domains under very general boundary conditions (Theorem 1). This theorem can be applied to yield uniqueness criteria as well as a priori bounds on solutions.

Our uniqueness results are n t entirely new; as was pointed out to us by Prof. P. Hartman, very similar criteria occur in Walter [10] and in Szarski [8]. Our theorem has the advantage of covering a large class of cylindrical domains with unbounded bases. Some of the ideas used in extending the theorem to unbounded domains can be modified to give a generalization of a well-known result of Tykhonov [9]; as the proof of this theorem is perhaps surprisingly easy, we include it in Section 2.

In Section 3, we discuss uniqueness theorems which may be obtained from Theorem 1, and also give some counter-examples, to show that non-uniqueness may occur in many non-linear problems. Section 4 is a survey of some of the applications of Theorem 1 to the problem of determining a priori bounds for solutions of (1). In order to show the existence of solutions of (1) in time intervals $0 \leq t \leq T$, with large T, it is necessary to obtain such a priori bounds in these intervals. This is done here by comparing the solution of (1) with the solution of (2), where $G(u,t)$ is chosen so as to majorize $F(x,t,u,\nabla u)$ in some appropriate sense, depending on the particular type of boundary value problem being considered. Fillipov [2] has recently (and independently) used a very similar argument to obtain a similar result in the case of the first mixed boundary value problem for bounded cylindrical

domains in two dimensions. Existence theorems for (1) also depend on obtaining a priori bounds on the gradient of the solution; for a discussion of this more difficult problem, we mention **Fillipov** [2] and, for the case of n dimensions, the excellent article of Oleinik and Kruzhkov [6].

We discuss briefly, in Section 5, some results in stability theory and the theory of the asymptotic behavior of solutions of (1). The main idea used here is due to Friedman [4]; in conjunction with Theorem 1, it enables us to give very simple proofs of several typical results.

Finally, in Section 6, we consider the question of obtaining estimates which could be used to show that solutions of (1) blow up in some finite time interval, if F grows too rapidly as a function of u. The sort of comparison argument used in Section 4 fails to yield this sort of information when the boundary values of the solution are prescribed. We are able, however, to show, for example, that any solution of the inequality

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) \geq F(u)$$

in a cylindrical domain with bounded base $\Omega \subset E^n$ can be made to blow up in any prescribed time interval, simply by making its initial values and/or boundary values large enough. Here,

(a_{ij}) is of course positive definite, and $F(u)$ is convex, and grows fast enough at infinity, i. e.,

$$\int^{+\infty} \frac{du}{F(u)} < +\infty$$

The method used here does not rely on the maximum principle but rather on a study of the ordinary differential inequality satisfied by an appropriate "Fourier coefficient" of u.

As most of this work is devoted to proving that certain inequalities imply certain others, and as all of the inequalities obtained may be reversed, and the new ones proved in the same way once the appropriate changes have been made in the inequalities appearing in the hypotheses, we choose not to spell this out on each occasion. This we point out in advance, so that our descriptions of certain results as "uniqueness theorems" or "stability theorems" etc. may make sense, even though we continually state and prove only one half of such theorems.

Some of the work which culminated in this article was done while the author was a Faculty of Arts and Sciences Fellow at Harvard University during the academic year 1960-61; the remainder was done during the academic year 1961-62, which the author spent as a Temporary Member of the Courant Institute of Mathematical Sciences. Several of the results described appear in the author's doctoral thesis submitted to Harvard in January, 1962.

1. A General Comparison Theorem

We are interested in equations of the form

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} = F(x,t,u,\nabla u). \qquad (1)$$

For the sake of simplicity, we consider (1) only in cylindrical domains $Q_T = \Omega \times (0,T]$, where Ω is an open, connected set in E^n; here $x = (x_1,\ldots,x_n)$, $u = u(x,t)$, and $\nabla u = (\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n})$. $F(x,t,u,p)$ is defined and continuous for all $(x,t) \in \bar{Q}_T$, $-\infty < u < +\infty$ and $p = (p_1,\ldots,p_n)$ where $-\infty < p_i < +\infty$, $i = 1,\ldots,n$. (a_{ij}) is defined and continuous on the same set, and is, in fact, symmetric and positive definite:

$$a_{ij}(x,t,u,p) = a_{ji}(x,t,u,p) \qquad i,j = 1,\ldots,n$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u,p)\, \xi_i \xi_j > 0 \quad \text{for all real}$$

$(\xi_1,\ldots,\xi_n) \neq 0$.

We shall be interested only in solutions $u(x,t)$ of class $C^{2,1}(\bar{Q}_T)$, by which we mean: u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $i,j = 1,\ldots,n$ are defined in $\Omega \times (0,T]$ and can be extended

to $\bar{\Omega} \times (0,T]$ so as to be continuous there. u is in fact continuous in all of $\bar{Q}_T$.

Throughout this work we assume that $\partial\Omega$, the boundary of Ω, is locally an n-1 dimensional C^{∞} manifold, with Ω always "on one side" of $\partial\Omega$. The C^{∞} assumption is certainly stronger than necessary. We shall call $\xi = \xi(x,t) = (\xi_1(x,t),\ldots,\xi_n(x,t))$ a (time-dependent) direction field on $\partial\Omega$, if, for each fixed t, $0 \leq t \leq T$, $\xi(x,t)$ is a function defined on some subset of $\partial\Omega$ with values on the unit n-sphere, i. e., the set of n-vectors of unit length. ξ is said to be exterior if $\xi(x,t) \cdot \eta(x) > 0$ for all x in the domain of ξ, $0 \leq t \leq T$, where $\eta(x)$ is the exterior normal to $\partial\Omega$ at x, and $\cdot$ indicates the ordinary Euclidean scalar product. ξ is called uniformly exterior if $\exists\ \delta > 0$ such that $\xi(x,t) \cdot \eta(x) \geq \delta$ for all x in the domain of ξ, $0 \leq t \leq T$. Given such a direction field and a differentiable function ϕ, we use the notation $\frac{\partial\phi}{\partial\xi}(x,t)$ to mean $\xi(x,t) \cdot \nabla\phi(x,t)$. In particular, $\frac{\partial\phi}{\partial\eta}(x,t)$ denotes the normal derivative of ϕ at (x,t).

We shall need, occasionally, a certain further assumption on $\partial\Omega$, which we call G1. For bounded Ω, a method due to Friedman [3] can be used to show that G1 is already satisfied under our previous hypotheses; for unbounded Ω, G1 seems to depend only on $\partial\Omega$ not wiggling too rapidly at infinity.

G1: $\exists$ a function $\rho(x)$, defined and twice continuously differentiable on Ω such that

a) $\nabla\rho(x)$ is in the direction $\eta(x)$ for every $x \in \partial\Omega$, i.e., $\nabla\rho(x) = \frac{\partial\rho}{\partial\eta}(x)\,\eta(x)$, and

$$\frac{\partial\rho}{\partial\eta}(x) \geq 1 \qquad \text{for all } x \in \partial\Omega.$$

b) $\exists$ $M > 0$ such that $|\rho(x)| \leq M$, $|\nabla\rho(x)| \leq M$, and

$$\left|\frac{\partial^2\rho}{\partial x_i \partial x_j}\right| \leq M \quad i,j = 1,\dots,n \quad \text{for all } x \in \Omega.$$

Given a uniformly exterior direction field $\xi(x,t)$, one observes that G1 has the following consequence, for unbounded domains Ω:

G2: $\exists$ a function $r(x)$, defined and twice continuously differentiable on Ω, such that:

a) $\frac{\partial r}{\partial \xi}(x) \geq 0$ for all x in the domain of ξ, $0 \leq t \leq T$.

b) $r(x) \sim |x|$ as $|x| \longrightarrow +\infty$, $x \in \bar{\Omega}$.

c) $\exists$ $M > 0$ such that $|\nabla r(x)| \leq M$ and

$$\left|\frac{\partial^2 r}{\partial x_i \partial x_j}\right| \leq M \quad i,j = 1,\dots,n, \quad \text{for all } x \in \bar{\Omega}.$$

To show that G2 follows from G1: Choose x_o, any point not on $\partial\Omega$, and $\psi(x)$, any C^∞ function which vanishes identi-

cally on some neighborhood N_o of x_o, and assumes the value one identically outside of some larger neighborhood of x_o, N_1, which does not intersect $\partial\Omega$. Then, take

$$r(x) = \psi(x)\ |x-x_o| + K\rho(x)$$

where K is any number $\geq \frac{1}{\delta}$, δ being the lower bound for $\xi \cdot \eta$ b) and c) are verified at once; a) follows if we observe that

$$\frac{\partial r}{\partial \xi} \geq K\ \xi \cdot \nabla\rho - 1 \geq K\delta - 1.$$

We observe that G2 holds trivially if $\partial\Omega$ is empty (i. e., $\Omega = E^n$, which corresponds to the so-called Cauchy problem for (1)) or if $\xi(x,t)$ is defined only on the empty set; (this corresponds, in what follows, to the case of the first boundary value problem for (1)).

Theorem 1, which we prove in this section, will have as its consequence a general uniqueness theorem, a special case of which will be a uniqueness theorem for the initial value problem for ordinary differential equations of the form

$$\frac{du}{dt} = G(t,u). \tag{2}$$

We recall a variant of a well-known uniqueness criterion for

(2); first, we need some definitions: If $\phi(t)$ is any real-valued function defined for $0 < t \leq T$, we define

$$\bar{\Phi}'(t) = \overline{\lim_{h \to 0+}} \ \frac{\phi(t) - \phi(t-h)}{h}.$$

We define R to be the class of all bounded, non-negative real-valued functions $\phi(t)$ defined for $0 \leq t \leq T$ such that

$$\overline{\lim_{h \to 0+}} \ \phi(t+h) \leq \phi(t) \qquad \text{for } 0 \leq t < T.$$

(Equivalently, we may define R as follows: $\phi \in R$ if and only if for every $\varepsilon \geq 0$, and every t_o with $0 \leq t_o < T$, the set $\{t \mid t \geq t_o, \ \phi(t) \geq \varepsilon\}$ contains its greatest lower bound.)

Finally, we say that the function $\psi(\phi, t)$ defined and continuous for $0 \leq t \leq T$, $0 \leq \phi < +\infty$ is of class U if the function $\phi(t) \equiv 0$ is the only function in R satisfying

$$\left.\begin{aligned} &\bar{\Phi}'(t) \leq \psi(\phi(t), t) \qquad 0 < t \leq T \\ \text{and} \quad & \\ &\phi(0) = 0 \end{aligned}\right\} \qquad (3)$$

Then, it is easy to prove the following: Let $u(t)$ and $v(t)$ be continuous in $[0,T]$, differentiable in $(0,T]$, and such that

$$\frac{du}{dt} \leq g(u(t),t) \qquad (2')$$

$$\frac{dv}{dt} \geq g(v(t),t) \qquad (2'')$$

Suppose $g(u,t) - g(v,t) \leq \psi_M(u-v,t)$ for $u > v$, $|u|, |v| \leq M$, and $0 < t \leq T$, where, for each $M > 0$, $\psi_M \in U$. Then, if $u(o) \leq v(o)$, it follows that

$$u(t) \leq v(t) \qquad \text{for } 0 \leq t \leq T.$$

Our comparison theorem for (1) will generalize this result. We need the following:

LEMMA. _If_ $\phi \in R$, _and_ $\bar{\phi}'(t) \leq 0$ _for_ $0 < t \leq T$, _then_ $\phi(t_1) \geq \phi(t_2)$ _for all_ $0 \leq t_1 \leq t_2 \leq T$.

Proof: Let $\alpha > 0$ be given; consider $\psi(t) = \phi(t) - \phi(t_1) - \alpha(t-t_1)$. Then $\psi \in R$, and $\bar{\psi}'(t) \leq -\alpha < 0$ for $0 < t \leq T$. Suppose $\psi(t) = \varepsilon > 0$ for some t, with $t_1 \leq t \leq T$; then there must be a least such t, say τ, and $t_1 < \tau \leq T$. But then, we must have $\bar{\psi}'(\tau) \geq 0$, and this is impossible. Thus, we must have

$$\psi(t) = \phi(t) - \phi(t_1) - \alpha(t-t_1) \leq 0 \quad \text{for all } t \geq t_1;$$

$\therefore$ $\phi(t_2) \leq \phi(t_1) + \alpha(t_2 - t_1)$, and since α is arbitrary, $\phi(t_2) \leq \phi(t_1)$.

This lemma, incidentally, allows us to prove the following useful and well-known criterion for membership in U:

If $\psi(\phi,t) \leq \lambda(t)\mu(\phi)$, where λ and μ are continuous,

$\int_0^T \lambda(t)dt < +\infty$, and, for every $\varepsilon > 0$, $\int_0^\varepsilon \frac{d\phi}{\mu(\phi)} = +\infty$,

then $\psi \in U$.

Since our definitions are mildly unorthodox, we include a proof of this assertion: Define $m(t) = \int_a^{\phi(t)} \frac{ds}{\mu(s)}$; we show that

$$\bar{m}'(t) \leq \lambda(t) \quad \text{for } 0 < t \leq T;$$

By the mean-value theorem, we have

$$m(t) - m(t-h) = \frac{1}{\mu(s^*)} [\phi(t) - \phi(t-h)]$$

where s^* lies between $\phi(t)$ and $\phi(t-h)$. Since $\bar{\dot{\phi}}'$ is bounded from above, by N, say, in $[t-h,t]$, our Lemma tells us that $\phi(t) - \phi(t-h) \leq Nh$; hence,

$$\underline{\lim}_{h \to 0+} \phi(t-h) \geq \phi(t).$$

We may thus assume that $\lim_{h \to 0+} \phi(t-h) = \phi(t)$, for, if not, we have $\bar{m}'(t) \leq 0 \leq \lambda(t)$. But, then, we obtain

$$\bar{m}'(t) = \frac{1}{\mu(\phi(t))} \bar{\phi}'(t) \leq \lambda(t).$$

To complete our proof, we invoke our Lemma; we conclude that $\int_0^{\phi(t)} \frac{ds}{\mu(s)} \leq \int_0^t \lambda(s)ds$ which implies that $\phi \equiv 0$.

We can now state the main result of this section:

THEOREM 1. Let u and v be two functions in $C^{2,1}(\bar{Q}_T)$ satisfying

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} \leq F(x,t,u,\nabla u) \qquad (1')$$

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,v,\nabla v) \frac{\partial^2 v}{\partial x_i \partial x_j} \geq F(x,t,v,\nabla v) \qquad (1'')$$

such that $u(x,0) \leq v(x,0)$ for $x \in \Omega$

and, for every $t \in (0,T]$, at every point $x \in \partial\Omega$ we have

either $\quad u(x,t) \leq v(x,t)$

or $\quad \frac{\partial u}{\partial \xi}(x,t) \leq \frac{\partial v}{\partial \xi}(x,t)$ where $\xi = \xi(x,t)$ is a given exterior direction field (for which G2 holds).*

*See remarks page 18.

We assume that for all $(x,t) \in \bar{Q}_T$, all u, v, p with $u > v$, $|u|, |v|, |p| \leq M$

$$F(x,t,u,p) - F(x,t,v,p) \leq \psi_M(u-v,t)$$

$$|a_{ij}(x,t,u,p) - a_{ij}(x,t,v,p)| \leq \psi_M(u-v,t) \quad i,j = 1,\ldots,n$$

where, for each K, $M > 0$, the function $K\psi_M(\phi,t) \in U$. (Also, F and the a_{ij} are assumed to be continuous in p; that continuity is uniform for $x \in \bar{\Omega}$ and for t, u, and p in compact sets. Similarly, a_{ij} is assumed to be uniformly bounded for $x \in \bar{\Omega}$ and for t, u, and p in compact sets.)* (U)

Finally, we suppose that u and v are bounded in $\bar{Q}_T$, and that at least one of u <u>or</u> v has first and second x-derivatives which are bounded in $\bar{Q}_T$.

Then we may conclude that $u \leq v$ in $\bar{Q}_T$.

Proof: Write $w(x,t) = u(x,t) - v(x,t)$; then

$$\frac{\partial w}{\partial t} \leq \sum_{i,j=1}^{n} \left[a_{ij}(x,t,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} - a_{ij}(x,t,v,\nabla v)\frac{\partial^2 v}{\partial x_i \partial x_j} \right]$$

$$+ F(x,t,u,\nabla u) - F(x,t,v,\nabla u)$$

Suppose it is v that has the bounded derivatives; then we rewrite the above as

$$\frac{\partial w}{\partial t} \leq L_u[w] + \sum_{i,j=1}^{n} [a_{ij}(x,t,u,\nabla u) - a_{ij}(x,t,v,\nabla v)]\frac{\partial^2 v}{\partial x_i \partial x_j}$$

$$+ F(x,t,u,\nabla u) - F(x,t,v,\nabla v)$$

$$\leq L_u[w] + K' \sum_{i,j=1}^{n} \Big\{ |a_{ij}(x,t,u,\nabla u) - a_{ij}(x,t,u,\nabla v)| \tag{4}$$

$$+ |a_{ij}(x,t,u,\nabla v) - a_{ij}(x,t,v,\nabla v)| \Big\}$$

$$+ F(x,t,u,\nabla u) - F(x,t,u,\nabla v) + F(x,t,u,\nabla v) - F(x,t,v,\nabla v)$$

where $L_u[w] = \sum_{i,j=1}^{n} a_{ij}(x,t,u,\nabla u)\frac{\partial^2 w}{\partial x_i \partial x_j}$.

(If it is u instead of v which has bounded derivatives, we get a similar expression with, for example, $L_v[w]$ replacing $L_u[w]$). We consider the function

$$\phi(t) = \max(\sup_{x \in \bar{\Omega}} w(x,t),\ 0).$$

Clearly $\phi(0) = 0$; we want to show that ϕ satisfies, for some K, $M > 0$, the inequality

$$\bar{\phi}'(t) \leq K\psi_M(\phi(t),t) \tag{5}$$

and also, that $\phi \in R$. This will enable us to conclude that

$\phi \equiv 0$, and hence our theorem will be proved.

1) Suppose Ω is bounded. Then, $\phi \in R$ since ϕ is in fact continuous in $[0,T]$. For any $t > 0$, if $\phi(t) = 0$, then (5) holds for any K, M, since in fact, $\Phi'(t) \leq 0$. If $\phi(t) > 0$, then $\exists\ x_0 \in \bar{\Omega}$ such that $w(x_0,t) = \phi(t)$. Since, for $h > 0$, $w(x_0,t-h) \leq \phi(t-h)$, we see that

$$\Phi'(t) \leq \lim_{h \to 0+} \frac{w(x_0,t) - w(x_0,t-h)}{h} = \frac{\partial w}{\partial t}(x_0,t).$$

If we can succeed in showing that at the point (x_0,t) we must have a) $\nabla w = 0$, i. e., $\nabla u = \nabla v$

and b) $L_u[w] \leq 0$

then (5) will follow from (4), if M is chosen so that

$$\sup_{\bar{Q}_T} |u(x,t)| \leq M, \quad \sup_{\bar{Q}_T} |\nabla u(x,t)| \leq M, \text{ and } \sup_{\bar{Q}_T} |\nabla v(x,t)| \leq M$$

and $K = nK' + 1$. But, if x_0 is an interior point of $\bar{\Omega}$, then a) and b) follow immediately by the well known maximum principle, since x_0 is a maximum point for w, and L_u is elliptic in a neighborhood of x_0. If $x_0 \in \partial\Omega$, then, since $w(x_0,t) = \phi(t) > 0$, we must have $\frac{\partial w}{\partial \xi}(x_0,t) \leq 0$. But, on the other hand, since x_0 is a maximum point for w in $\bar{\Omega}$, $\nabla w(x_0,t)$ is a vector (possibly null) pointed in the direction of the exterior normal to $\partial\Omega$ at x_0. Thus, $\nabla w(x_0,t) = 0$; otherwise,

we would have $\frac{\partial w}{\partial \xi}(x_o,t) > 0$, since ξ is an exterior direction. If $L_u[w] > 0$ at (x_o,t), then $L_u[w] > 0$ in a neighborhood of x_o in $\bar{\Omega}$; hence, by E. Hopf's extension of the maximum principle for elliptic operators [5] $\nabla w(x_o,t)$ would be non-zero Thus, b) also follows.

2) If Ω is not bounded: Our problem here is complicated by the fact that w need not assume its maximum; to avoid this difficulty, we approximate w by something which must. Define, for $\varepsilon > 0$

$$w_\varepsilon(x,t) = w(x,t) - \varepsilon r(x) \qquad (r(x) \text{ comes from G2});$$

$$\phi_\varepsilon(t) = \max\left(\sup_{x \in \bar{\Omega}} w_\varepsilon(x,t), 0\right)$$

We shall first show that

$$\bar{\phi}'_\varepsilon(t) \leq K\psi_M(\phi(t),t) + \alpha(\varepsilon) \tag{6}$$

where K and M are as above, and $\alpha(\varepsilon) \dashrightarrow 0$ as $\varepsilon \dashrightarrow 0$: Let $\varepsilon > 0$ be given, along with some t, $0 < t \leq T$; if $\phi_\varepsilon(t) = 0$, (6) holds trivially. If $\phi_\varepsilon(t) > 0$, then $\exists\, x_o \in \bar{\Omega}$ such that $\phi_\varepsilon(t) = w_\varepsilon(x_o,t)$. As before, $\bar{\phi}'_\varepsilon(t) \leq \frac{\partial w_\varepsilon}{\partial t}(x_o,t) = \frac{\partial w}{\partial t}(x_o,t)$. At (x_o,t), it follows by the arguments already used, that

a) $\nabla w_\varepsilon = 0$, i. e., $\nabla u - \nabla v = \varepsilon\nabla r$

and b) $L_u[w_\varepsilon] \leq 0$, i. e., $L_u[w] \leq \varepsilon L_u[r]$.

(We use here the fact that $\frac{\partial w_\varepsilon}{\partial \xi} = \frac{\partial w}{\partial \xi} - \varepsilon\frac{\partial r}{\partial \xi} \leq \frac{\partial w}{\partial \xi}$.) (6) then follows immediately from (4), our assumption (U), and the properties of $r(x)$. We now deduce (5) from (6): (6) implies that $\exists$ $N > 0$, independent of ε (for $\varepsilon \leq 1$, say) such that $\bar{\Phi}'_\varepsilon(t) \leq N$ for $0 < t \leq T$. Since $\phi_\varepsilon \in R$, (it is, in fact, continuous), our Lemma allows us to conclude that

$$\phi_\varepsilon(t_2) - \phi_\varepsilon(t_1) \leq N(t_2-t_1) \quad \text{for } 0 \leq t_1 \leq t_2 \leq T.$$

Clearly, $\phi_\varepsilon(t)$ converges pointwise to $\phi(t)$ as $\varepsilon \to 0$.

$$\therefore \quad \phi(t_2) - \phi(t_1) \leq N(t_2-t_1) \quad \text{for } 0 \leq t_1 \leq t_2 \leq T.$$

From this, we conclude that $\phi \in R$, and, in addition

$$\underline{\lim}_{h \to 0+} \phi(t-h) \geq \phi(t) \quad \text{for } 0 < t \leq T.$$

If $\overline{\lim}_{h \to 0+} \phi(t-h) > \phi(t)$, then $\bar{\Phi}'(t) \leq 0$, and (5) holds trivially. Thus, we may assume that $\lim_{h \to 0+} \phi(t-h) = \phi(t)$, for $0 < t \leq T$. Going back to the argument used to show that

$$\phi(t_2) - \phi(t_1) \le N(t_2 - t_1)$$

we see that N may be taken as:

$$K \sup_{t_1 \le t \le t_2} \psi_M(\phi(t),t) \dashrightarrow K\psi_M(\phi(t_2),t_2)$$

as t_1 approaches t_2 from below. Thus, we obtain (5), and our proof is complete.

Remarks. *In our proof, we needed the hypotheses marked with an asterisk only for the case of unbounded Ω.

If it should happen (as it does in several of our applications below) that $\frac{\partial^2 v}{\partial x_i \partial x_j} = 0 \quad i,j = 1,\ldots,n$, we may drop all of the hypotheses (U) pertaining to the a_{ij} except for the uniform boundedness assumption. Also, in this case, we need only assume that $\psi_M(\phi,t) \in U$ for all $M > 0$.

To see that Theorem 1 really generalizes the result mentioned above, observe that we may consider a solution of (2) as a solution of (1) with $F(x,t,u,p) \equiv G(t,u)$, any Ω, and suitable ξ.

2. A generalization of a theorem of Tykhonov

Our assumptions (U) on F and the a_{ij} in Theorem 1 are just local in nature, in so far as dependence on u and p is con-

cerned; our theorem is saved by the fact that we assume a priori that our solutions are bounded in $\bar{\Omega}$, for all t in $[0,T]$. Does anything like Theorem 1 hold with any relaxation of this boundedness requirement? For example, Tykhonov [9] proved uniqueness in the Cauchy problem for the heat equation under the assumption that the solutions in question grow no faster than $e^{\alpha|x|^2}$ for some $\alpha > 0$; it is this result which one might wish to generalize. We are unable, however, to do anything with arbitrary F, but must content ourselves with the case where F grows only linearly. Thus, our result is not completely new, but it is perhaps surprising that it can be obtained in such generality with so little effort, and, in particular, without the use of a fundamental solution.

THEOREM 2. <u>Let an unbounded Ω, with an exterior direction field ξ satisfying</u> G2 <u>be given; let</u> $u \in C^{2,1}(\bar{Q}_T)$ <u>satisfy</u>

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} \leq F(x,t,u,\nabla u) \qquad (7)$$

where (a_{ij}) is positive definite in $\bar{Q}_T$ (not necessarily uniformly). Suppose $\exists\ M > 0$ such that $|a_{ij}(x,t)| \leq M$ $i,j = 1,\ldots, n$ for $(x,t) \in \bar{Q}_T$ and

$$F(x,t,u,p) \leq m_2(x)u + m_1(x)|p|$$

for all $u \geq 0$, all p, and all $(x,t) \in \bar{Q}_T$, where

$$m_1(x) \leq M(1+|x|) \text{ and } m_2(x) \leq M(1+|x|)^2.$$

Then, if $u(x,0) \leq 0$ for all $x \in \Omega$, and if, for every t, $0 < t \leq T$, we have at each $x \in \partial\Omega$.

either $\quad u(x,t) \leq 0$

or $\quad \frac{\partial u}{\partial \xi}(x,t) \leq 0$

and if, in addition, $\exists$ α, $K > 0$ such that

$$u(x,t) \leq Ke^{\alpha|x|^2} \qquad \text{for } (x,t) \in \bar{Q}_T,$$

then we may conclude that

$$u(x,t) \leq 0 \qquad \text{for all } (x,t) \in \bar{Q}_T.$$

Proof: We shall prove our assertion first in $\bar{Q}_\delta$ for some $\delta > 0$ depending only on Ω, α, and M; by replacing t by $t-\delta$, we extend our assertion to $\bar{Q}_{2\delta}$. and, repeating this often enough, our theorem will be proved. We observe that if $v(x,t)$ satisfies, in Q_δ, the inequality

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i \partial x_j} \leq G(x,t,v,\nabla v) \tag{8}$$

where $G(x,t,v,0) \leq 0$ for $v > 0$, and if v satisfies the same initial and boundary conditions as u, and, in addition,

$$\overline{\lim_{|x|\to\infty}}\; v(x,t) = 0 \quad \text{uniformly in } [0,\delta]$$

then, the reasoning used in part 1) of the proof of Theorem 1 sufficies to show that $v(x,t) \leq 0$ for all $(x,t) \in \bar{Q}_\delta$. Thus,

we define $v(x,t) = u(x,t)\, e^{-\lambda(t)(r(x)+\beta)^2}$, where $\lambda(t)$ and β will be specified below. We demand of β only that it be large enough so that

$$r(x) + \beta \geq 1 \quad \text{in } \bar{\Omega}.$$

Then, $\exists\, N_1 > 0$ such that $1 + |x| \leq N_1(r(x)+\beta)$ in $\bar{\Omega}$. Clearly, $v(x,0) \leq 0$ for all $x \in \bar{\Omega}$. Since

$$\frac{\partial v}{\partial \xi} = e^{-\lambda(t)(r+\beta)^2}\frac{\partial u}{\partial \xi} - 2\lambda(t)(r+\beta)v\frac{\partial r}{\partial \xi}$$

we may also conclude that v satisfies the same conditions u does at $\partial\Omega$. ($\lambda(t)$ is to be chosen $> \alpha$ for $0 \leq t \leq \delta$, of course.)

We have

$$\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = \left(\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) e^{-\lambda(r+\beta)^2}$$

$$- \lambda'(t)(r+\beta)^2 v$$

$$+ 2\lambda(r+\beta) e^{-\lambda(r+\beta)^2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial r}{\partial x_j}$$

$$+ 2\lambda v \sum_{i,j=1}^{n} a_{ij} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j}$$

$$+ 2\lambda(r+\beta) \sum_{i,j=1}^{n} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial r}{\partial x_j}$$

$$+ 2\lambda(r+\beta) v \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 r}{\partial x_i \partial x_j} .$$

When $\nabla v = 0$, $\nabla u = 2\lambda(r+\beta) v\, e^{\lambda(r+\beta)^2} \nabla r$; thus, v satisfies (8), where, for $\nabla v = 0$, and $v > 0$ (i. e., $u > 0$),

$$G(x,t,v,0) \leq [N(\lambda^2(t)+\lambda(t)+1) - \lambda'(t)](r+\beta)^2 v$$

where N depends only on [illegible], M, and α. Thus, $v(x,t) \leq 0$ in $\bar{Q}_\delta$, (and hence $u(x,t) \leq 0$ in $\bar{Q}_\delta$) provided that λ satisfies

$$\lambda'(t) \geq N(\lambda^2(t)+\lambda(t)+1) \quad \text{for } 0 \leq t \leq \delta$$

and $\lambda(0) > \alpha$. But such a λ exists, for $\delta > 0$ sufficiently small, depending only on N and α. This completes our proof.

3. Uniqueness theorems and counter-examples

Theorem 1 is easily seen to imply uniqueness for a variety of mixed initial-boundary value problems for (1), for example, problems where a solution u(x,t) of (1) is sought under the following conditions:

$$u(x,0) \text{ is prescribed for } x \in \bar{\Omega},$$

and 1) u(x,t) is prescribed for $x \in \partial\Omega$, and $0 < t \leq T$,

or 2) $\frac{\partial u}{\partial \xi}(x,t)$ is prescribed for $x \in \partial\Omega$, and $0 < t \leq T$, where ξ is some exterior direction field (satisfying G2, if Ω is unbounded),

or more generally

3) $a(x,t)u + b(x,t)\frac{\partial u}{\partial \xi}$ is prescribed for $x \in \partial\Omega$, and $0 < t \leq T$, where a and b are given non-negative functions, and $a + b > 0$ for $x \in \partial\Omega$, $0 < t \leq T$, (ξ is as in 2)),

or even

4) $g(x,t,u,\nabla u)$ is prescribed for $x \in \partial\Omega$, and $0 < t \leq T$, where g(x,t,u,p) is non-decreasing when considered

as a function of u and as a function of $\xi \cdot p$, and is strictly increasing in at least one of these, (ξ as in 2) and 3)), for fixed (x,t), $x \in \partial \Omega$ and $0 < t \leq T$.

Since our boundary assumptions in Theorem 1 hold for any two solutions u and v of problems 1) - 4), our assumptions (U) on F and the a_{ij} guarantee uniqueness for these problems. Suppose, however, the assumptions (U) are not made; more specifically, suppose we assume nothing about F. As we have seen, counter-examples to uniqueness for the initial-value problem for (2), in the absence of the appropriate restrictions on $G(u,t)$, are also counter-examples to uniqueness in problem 2) above (and, in the Cauchy problem for (1) also). As we need it below, we mention a well-known class of such counter-examples: The equation

$$\frac{d\psi}{dt} = \psi^{\alpha} \quad (0 < \alpha < 1) \tag{9}$$

with $\psi(0) = 0$

has the family of solutions

$$\psi(t) = \begin{cases} 0 \quad \text{for } 0 \leq t \leq \beta \\ [(1-\alpha)(t-\beta)]^{1/1-\alpha} \text{ for } t \geq \beta \end{cases}$$

But, suppose our problem involves the prescription of boundary values of the solution; then, the above examples no longer suffice to refute uniqueness. However, we can find new counter-examples, at least for equations of the special form

$$\frac{\partial u}{\partial t} - L[u] = F(x,t,u,\nabla u)$$

where L is the self-adjoint, uniformly elliptic operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) .$$

(Uniform ellipticity means, of course, the existence of a constant $m > 0$, called the ellipticity constant, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i\xi_j \geq m \sum_{i=1}^{n} \xi_i^2$$

for all $x \in \bar{\Omega}$, all real $\xi_1,\ldots,\xi_n$.) Then we consider the following eigenvalue problem:

to find a regular (i. e., C^2) solution $\phi(x)$ of

$$-L\ [\phi] = \lambda\phi \qquad \text{in } \Omega \tag{10}$$

$$\text{with} \qquad a(x) \quad + b\frac{\partial\phi}{\partial\xi} = 0 \qquad \text{on } \partial\Omega \tag{11}$$

where $a(x)$ and $b(x)$ are as in problem 3) above, and $\xi = \xi(x)$ is a time-independent exterior direction field on $\partial\Omega$. If $a(t)$ is any function defined in $(0,T]$ such that

$$\int_0^{\varepsilon} a(t)dt = +\infty \text{ for every } \varepsilon > 0 \text{ (for example, } a(t) = \frac{1}{t}\text{)}$$

the equation

$$\frac{d\psi}{dt} = a(t)\,\psi \tag{12}$$

has an infinity of different solutions satisfying

$$\psi(0) = 0;$$

hence the equation

$$\frac{\partial u}{\partial t} - L[u] = (\lambda + a(t))u$$

with the conditions $u = 0$ for $t = 0$

and $au + b\frac{\partial u}{\partial \xi} = 0$ for $x \in \partial\Omega$.

has the infinity of solutions

$$u(x,t) = \phi(x)\,\psi(t), \quad \text{where}$$

ϕ satisfies (10) and (11), and ψ satisfies (12), with $\psi(0) = 0$. (This class of counter-examples is due, at least in a special case, to Westphal [11].) One may raise the objection that $F(x,t,u,p)$ is not, in this case, a well-behaved function of t as t --> 0. To meet this objection, we suppose that a solution $\phi(x)$ of (10) and (11) exists which is non-negative in Ω (i. e., a solution exists which does not change sign there, since $-\phi$ is always a solution along with ϕ). Then, if $\psi(t)$ is any solution of (9), with $\psi(0) = 0$, the function

$$u = \psi(t)\ \phi(x)$$

satisfies $$\frac{\partial u}{\partial t} - L[u] = \lambda u + \phi^{1-\alpha} u^{\alpha}$$

with $$u = 0 \quad \text{for } t = 0$$

and $$au + b\frac{\partial u}{\partial \xi} = 0 \quad \text{for } x \in \partial\Omega.$$

To complete this discussion, we should say something about the existence of the eigenfunctions used above. Much of the theory of such problems, for bounded Ω, and for the special case in which $\xi = \eta$, i. e., the classical mixed Dirichlet-Neumann problem, is contained in [1], pp. 397-414, where it is shown that solutions ϕ exist for an increasing discrete set of

eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \longrightarrow +\infty$$

and that the zeros of ϕ_m, the eigenfunction corresponding to the eigenvalue λ_m, can divide Ω into no more than m components. (This latter assertion we call Courant's Theorem; we use it again in Section 6, below.) In particular, ϕ_1 cannot change sign in $\tilde{\Omega}$. The proof given in [1] is not complete, in the generality desired. The most important missing part is the proof that the so-called "weak solutions" found by variational techniques are really regular at $\partial\Omega$. This can be done, however, at least for the case of pure Dirichlet data, i. e. $a \equiv 1$, $b \equiv 0$, and also in the case where a and b are both smooth functions, and b never vanishes, so that we may rewrite our boundary conditions as

$$\frac{\partial u}{\partial \xi} + c(x)\, u = 0 \qquad \text{on } \partial\Omega.$$

(see [7], for example).

Having completed our discussion of counter-examples, we stipulate that in the sequel, F and the a_{ij} are always assumed to satisfy the hypotheses (U) in Theorem 1.

4. A priori bounds on solutions

To illustrate how solutions of (1) may be bounded a priori by means of Theorem 1, we consider first the Cauchy problem, i. e., $u(x,t)$ is a bounded regular solution of (1) in $Q_T = \Omega \times (0,T]$ where $\Omega = E^n$. Suppose

$$\sup_{x \in E^n} F(x,t,u,0) \leq G(t,u) \qquad \text{for } 0 < t \leq T, \text{ all } u.$$

If G is such that the equation

$$\frac{du}{dt} = G(t,u) \tag{2}$$

has a solution $\phi(t)$, differentiable in $(0,T]$ and satisfying (2) there, and continuous in $[0\ T]$, with

$$\phi(0) \geq \sup_{x \in E^n} u(x,0)$$

then, by Theorem 1,

$$\sup_{x \in E^n} u(x,t) \leq \phi(t) \quad \text{for } 0 \leq t \leq T.$$

In particular, we see that if

$$\sup_{x \in E^n} F(x,t,u,0) \leq \lambda(t)\mu(u) \quad \text{for } 0 < t \leq T, \text{ all } u,$$

where $$\int_0^T \lambda(t)dt < \int_a^{+\infty} \frac{du}{\mu(u)},$$

then if $u(x,t)$ is a bounded solution of (1) in some Q_{τ}, $0<\tau\leq T$, with $\sup_{x \in E^n} u(x,0) \leq a$, then u is bounded from above by a constant independent of τ. If this is to be true for all a, and $\lambda(t) > 0$, then we must have $\int_a^{+\infty} \frac{du}{\mu(u)} = +\infty$ for all a.

Obviously, similar considerations apply in obtaining a priori bounds from below. In fact, in this case, because of the absence of boundary conditions, one may apply the same reasoning to give a criterion for the "blowing up" of the solution:

If $$\inf_{x \in E^n} F(x,t,u,0) \geq G(t,u) \text{ for } 0 < t \leq T, \text{ all } u$$

where G is such that (2) has a solution $\psi(t)$ with

$\psi(0) \leq \inf_{x \in E^n} u(x,0)$, such that $\psi(t) \dashrightarrow +\infty$ as $t \dashrightarrow T$,

then $u(x,t)$ can not be a bounded solution of (1) in Q_T; in fact, if u is such a solution in Q_{T_0}, then

$$\inf_{x \in E^n} u(x,t) \geq \psi(t) \text{ for } 0 \leq t \leq T_0.$$

For more general initial-boundary value problems we must modify our reasoning. Consider the first mixed boundary value problem. Suppose

$$\sup_{x \in \Omega} F(x,t,u,0) \leq G(t,u) \text{ for } 0 < t \leq T, \text{ all } u,$$

where G is such that (2) has a solution $\phi(t)$ in $(0, T]$

$$\text{with} \qquad \phi(0) \geq \sup_{x \in \Omega} u(x,0) \tag{a}$$

$$\text{and} \qquad \phi(t) \geq \sup_{x \in \partial\Omega} u(x,t) \text{ for } 0 < t \leq T; \tag{b}$$

here $u(x,t)$ is a bounded solution of (1) in Q_T. Then, by Theorem 1, $\sup_{x \in \bar{\Omega}} u(x,t) \leq \phi(t)$ for $0 \leq t \leq T$. (a) and (b) are rather unwieldy conditions; if G can be chosen non-negative, then it is sufficient to assume that ϕ satisfies

$$\phi(0) \geq \max \left(\sup_{x \in \Omega} u(x,0), \sup_{\substack{x \in \partial\Omega \\ 0<t\leq T}} u(x,t) \right) \tag{c}$$

from which (a) and (b) follow. Since, for the first mixed boundary value problem, the expression on the right hand side of (c) is known in advance, this criterion is a useful one. Similarly, to get bounds from below, if we assume

$$\inf_{x \in \Omega} F(x,t,u,0) \geq G(t,u)$$

where G, now assumed to be non-positive, is such that (2) has a solution $\psi(t)$ satisfying

$$\psi(0) \leq \min \left(\inf_{x \in \Omega} u(x,0),\ \inf_{x \in \partial\Omega} u(x,t) \right)$$

then $u(x,t) \geq \psi(t)$ for $0 \leq t \leq T$.

It is impossible to apply Theorem 1 in this way to show that a solution $u(x,t)$ of the first mixed boundary value problem for (1) "blows up" in a finite time: for, if $\phi(t)$ is to bound $u(x,t)$ from below, with $\phi(t)$ approaching $+\infty$ as $t \to T_0$, then clearly u must become infinite on $\partial\Omega$ as $t \to T_0$, if Theorem 1 is to be applicable. But the boundary values of u are prescribed, and well-behaved. We shall return to this question in Section 6, where we employ another method to prove the existence of "finite escape times".

In the special case of bounded Ω, we may generalize the previous result:

THEOREM 3. Let $u(x,t)$ be a solution of (1) in Q_T. Suppose for some fixed $p^0 = (p_1^0,\dots,p_n^0)$ we have

$$\sup_{x \in \Omega} F(x,t,u,p^0) \leq G(t,u) \qquad \text{for } 0 < t \leq T, \text{ all } u,$$

where $G(t,u)$ is non-negative, and non-decreasing in u for each fixed t. Let d (finite) be the diameter of $\bar{\Omega}$ (i. e., d is the maximum distance between any two points of $\bar{\Omega}$.) Then, if $\phi(t)$ is a solution of (2) in $(0,T]$

with $$\phi(0) \geq \max \left(\sup_{x \in \Omega} u(x,0), \sup_{\substack{x \in \partial\Omega \\ 0<t\leq T}} u(x,t) \right) + 2d|p^o|$$

we have $$\sup_{x \in \Omega} u(x,t) \leq \phi(t) \qquad \text{for } 0 \leq t \leq T.$$

Proof: Write $v(x,t) = \phi(t) + p^o \cdot (x-x_o) - d|p^o|$ where $x^o = (x_1^o,\ldots,x_n^o)$ is any fixed point in Ω.

Then
$$\begin{aligned}\frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,v,\nabla v)\frac{\partial^2 v}{\partial x_i \partial x_j} &= \frac{d\phi}{dt} \\ &= G(t,\phi) \\ &\geq G(t,v) \\ &\geq F(x,t,v,\nabla v)\end{aligned}$$

since $\phi \geq v$, $\nabla v = p^o$, and $\frac{\partial^2 v}{\partial x_i \partial x_j} = 0$ for $i,j = 1,\ldots n$.

Since ϕ was chosen to make v dominate u when $t = 0$ and when $x \in \partial\Omega$, we may apply Theorem 1, to complete our proof.

In the above, we have been able to ignore completely the growth of the a_{ij} as functions of u and p. In attempting to

apply similar techniques to more general boundary value problems, however, we can no longer find dominating functions $v(x,t)$ whose second partial derivatives (with respect to x) vanish identically. But we may still apply Theorem 1, as the following result shows:

THEOREM 4. *Let* $u(x,t)$ *be a bounded solution of (1) in* Q_T. *We assume that G1 holds for* Ω. *Let* M *and* $\rho(x)$ *be as in the statement of G1, let* δ *be the lower bound for* $\xi \cdot \eta$, *where* $\xi(x,t)$ *is a given uniformly exterior direction field, and suppose that* $\exists\, N > 0$ *such that*

a) $\sup\limits_{x \in \Omega} u(x,0) \leq N$

and

b) *for every* t *with* $0 < t \leq T$; *at each* $x \in \partial\Omega$,
either $u(x,t) \leq N$
or else $\frac{\partial u}{\partial \xi}(x,t) \leq N$.

Write $R = \frac{MN}{\delta}$. *We assume that*

$$\sup_{\substack{x \in \bar{\Omega} \\ |p| \leq R}} |a_{ij}(x,t,u,p)| \leq H(t,u) \quad \text{for } 0 < t \leq T,$$

all u, *and* $i, j = 1,\ldots,n$, *and that*

$$\sup_{\substack{x \in \Omega \\ |p| \leq R}} F(x,t,u,p) \leq G(t,u) \text{ for } 0 < t \leq T, \text{ all } u,$$

where $H(t,u)$ *and* $G(t,u)$ *are both non-negative, and non-decreasing in* u *for each fixed* t. *Then, if* $\phi(t)$ *is a solution in* $(0,T]$ *of*

$$\phi'(t) = G(t,\phi) + n\,R\,H(t,\phi)$$

with $\quad \phi(0) \geq N + 2R$

we have $\quad \sup_{x \in \Omega} u(x,t) \leq \phi(t) \quad$ *for* $0 \leq t \leq T$.

Proof: Take $v(x,t) = \phi(t) + \frac{N}{\delta}\rho(x) - R$. Then, $v(x,0) \geq N \geq u(x,0)$ for all $x \in \bar{\Omega}$. For $0 < t \leq T$, and any $x \in \Omega$, if $u(x,t) \leq N$, then $u(x,t) \leq v(x,t)$, since $v(x,t) \geq \phi(t) - 2R \geq N$; if $\frac{\partial u}{\partial \xi}(x,t) \leq N$, then $\frac{\partial u}{\partial \xi}(x,t) \leq \frac{\partial v}{\partial \xi}(x,t)$, since

$$\frac{\partial v}{\partial \xi}(x,t) = \frac{N}{\delta}\,\xi(x,t).\ \nabla\rho(x) \geq N.$$

Since $\quad v \leq \phi$, and $|\nabla v| = \frac{N}{\delta}\,|\nabla\rho| \leq R$, we have

$$\begin{aligned} \frac{\partial v}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t,v,\ v)\frac{\partial^2 v}{\partial x_i \partial x_j} &\geq \phi'(t) - n\,R\,H(t,v) \\ &\geq G(t,\phi) + n\,R\,H(t,\phi) \\ &\qquad - n\,R\,H(t,v) \\ &\geq G(t,v) \\ &\geq F(x,t,v,\nabla v). \end{aligned}$$

Applying Theorem 1, our proof is complete.

To answer this, we must be able to bound $\sup_{x \in \bar{\Omega}} u(x,t)$ from below; this we do in the following theorem.

THEOREM 8. We suppose that Ω is bounded, and that $u(x,t) \in C^{2,1}$ in Q_T, and satisfies there

$$\frac{\partial u}{\partial t} - L[u] \geq G(u,t) \tag{15}$$

where $L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j})$ is a self-adjoint uniformly elliptic differential operator with smooth coefficients (in $C^3(\bar{\Omega})$, say) and $G(u,t)$ is a convex function of u for each fixed $t \geq 0$. Let $\phi(t)$ satisfy

$$\frac{d\phi}{dt} = G(\phi,t) - \lambda_1(\phi - k(t)) \qquad \text{for } 0 < t \leq T,$$

and

$$\phi(0) = \inf_{x \in \bar{\Omega}} u(x,0),$$

where $k(t) = \inf_{x \in \partial\Omega} u(x,t)$, and where λ_1 is the first eigenvalue for the problem

$$-L[\psi] = \lambda\psi \qquad \text{in } \Omega$$

with

$$\psi = 0 \qquad \text{on } \partial\Omega.$$

Then, we have $\sup_{x \in \bar{\Omega}} u(x,t) \geq \phi(t)$ for $0 \leq t \leq T$.

Proof: By Courant's Theorem, the eigenfunction $\psi(x)$ corresponding to λ_1, does not change sign in Ω. We may thus

5. Asymptotic behavior and stability

In obtaining the results of the previous section, we were content, roughly speaking, to exploit the fact that the elliptic part of our parabolic operator could not hurt us too much; in this section we attempt to exploit the fact that the elliptic part can actually help in obtaining a priori bounds. To that end, we assume that all of the differential operators of the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j}$$

which figure in what follows are uniformly elliptic in $\bar{Q}_\infty$ $[= \bar{\Omega} \times [0,\infty)]$ with ellipticity constant $\geq m$. Here, we assume only that Ω is bounded in at least one direction, i. e., $\exists$ two parallel n-1 dimensional hyperplanes such that Ω is contained in the slab between them. We may assume that these hyperplanes are given by the equations $x_1 = 0$ and $x_1 = h$, if necessary performing a rigid motion in order to bring this about. We need the following lemma, which is proved implicitly by Friedman [4] in even greater generality:

LEMMA. <u>Given</u> $C > 0$, $\exists$ $\beta_o > 0$ and $\phi(x) \in C^2(\bar{\Omega})$, with $\phi(x) \geq 1$ in $\bar{\Omega}$, depending only on C, m, and h, such that

$$-L[\phi] \geq 1 + \beta_o \phi + C|\nabla\phi| \quad \text{in } \Omega ,$$

for every L.

Proof: We look for a function of the form $\phi(x) = e^{\lambda R} - e^{\lambda x_1}$ where λ and R are positive constants to be determined. Since

$$-L[\phi] \geq m\lambda^2 e^{\lambda x_1}$$

we try to choose λ and R so that

$$m\lambda^2 e^{\lambda x_1} \geq 1 + \beta_o(e^{\lambda R} - e^{\lambda x_1}) + C\lambda e^{\lambda x_1}.$$

First we choose λ large enough so that

$$m\lambda^2 > C\lambda + 1 ;$$

then we choose R large enough so that

$$e^{\lambda R} - e^{\lambda h} \geq 1.$$

Finally, if we take

$$\beta_o = \frac{m\lambda^2 - C\lambda - 1}{e^{\lambda R} - 1}$$

we satisfy all of the conditions necessary to complete our proof.

For use below, we write

$$A = \sup_{x \in \Omega} \phi(x) \quad , \quad B = \sup_{x \in \Omega} |\nabla\phi(x)|.$$

We deal here only with the first mixed boundary value problem. None of the results of this section are true for problem 2) of Section 2, as is seen by considering solutions u which depend only on t, not on x. In order to extend the present method to more general boundary value problems, one would have to generalize our Lemma in the obvious direction. We content ourselves, however, with a few sample results, which are of interest here mainly because their proofs are so trivial, in the present context.

The first theorem in this section is essentially a special case of Theorem 1 in [4]; the proof we give is merely a paraphrase of that in [4], shortened somewhat. Let us remark that since we do not assume that Ω is bounded, and since we intend to apply Theorem 1, we must assume that the solutions $u(x,t)$ which appear below are bounded in each $\bar{Q}_T$, $0 < T < +\infty$ (the bound may depend on T, of course). In addition, we assume, as always, that $u \in C^{2,1}$, this time in Q_∞.

THEOREM 5. <u>Suppose</u> $u(x,t)$ <u>satisfies, in</u> Q_∞,

$$\frac{\partial u}{\partial t} - L[u] \leq F(x,t,u,\nabla u) \tag{13}$$

where $$F(x,t,u,p) \leq \alpha(x,t) + \beta(x,t)u + C|p| \qquad (14)$$

for $u \geq 0$, $x \in \bar{\Omega}$, $0 \leq t \leq \infty$,

and all p.

Suppose further that

$$\overline{\lim_{t \to +\infty}} \left(\sup_{x \in \Omega} \alpha(x,t) \right) \leq 0$$

and $$\overline{\lim_{t \to +\infty}} \left(\sup_{x \in \Omega} \beta(x,t) \right) < \beta_0$$ (β_0 *as in our Lemma, depending on* C.) *Then, if*

$$\overline{\lim_{t \to +\infty}} \left(\sup_{x \in \partial\Omega} u(x,t) \right) \leq 0$$

it follows that

$$\overline{\lim_{t \to +\infty}} \left(\sup_{x \in \Omega} u(x,t) \right) \leq 0.$$

Proof: Given $\varepsilon > 0$, we may choose $T_0 > 0$ large enough so that for $t \geq T_0$ we have $\sup_{x \in \bar{\Omega}} \alpha(x,t) \leq \varepsilon$

$$\sup_{x \in \bar{\Omega}} \beta(x,t) \leq \beta_0$$

and $$\sup_{x \in \partial\Omega} u(x,t) \leq \varepsilon.$$

Let $M_o = \sup_{x \in \bar{\Omega}} u(x,T_o)$. Then, write

$$v(x,t) = (2\varepsilon + M_o e^{\varepsilon/AM_o (T_o-t)}) \quad \phi(x);$$

we wish to apply Theorem 1, but this time in the cylinder $\Omega \times (T_o, \infty)$, i. e., in the cylinder $\Omega \times (T_o, T]$, for arbitrarily large T: We observe that for all $x \in \bar{\Omega}$, we have

$$v(x,T_o) \geq M_o \geq u(x,T_o); \text{ also,}$$

$$v(x,t) \geq 2\varepsilon \geq u(x,t) \quad \text{for } x \in \partial\,\Omega,\ t > T_o.$$

Finally, we observe that for $t > T_o$,

$$\begin{aligned} \frac{\partial v}{\partial t} - L[v] &\geq -\frac{\varepsilon}{A} \quad \phi(x) + 2\varepsilon \quad + \beta_o v + C|\nabla v| \\ &\geq \varepsilon + \beta_o v + C|\nabla v| \\ &\geq \alpha(x,t) + \beta(x,t)v + C|\nabla v| \\ &\geq F(x,t,v,\nabla v). \end{aligned}$$

Thus, by Theorem 1, $u(x,t) \leq v(x,t)$ for $t \geq T_o$, $x \in \bar{\Omega}$; but

$$\overline{\lim_{t \to 0+}} \left(\sup_{x \in \bar{\Omega}} v(x,t) \right) = 2\varepsilon A,$$

and since ε was arbitrary, our proof is complete.

Remark. Suppose we know a priori that $u(x,t) \leq M$ in $\bar{Q}_\infty$; then, in order to prove Theorem 5, we need assume that (14) holds only for $0 \leq u \leq AM + \delta$ and $0 \leq |p| \leq BM + \delta$, for some $\delta > 0$. For, since $v(x,t) \leq AM + \delta$, and $|\nabla v(x,t)| \leq BM + \delta$ (provided that ε is small enough) our proof goes through unchanged. Armed with this observation, we next attack a "stability theorem" in which all we need know is the behavior of F for small u and $|p|$.

THEOREM 6. Suppose $u(x,t)$ satisfies (13) in Q_∞, where $F(x,t,u,p) \leq (\beta_0 + \frac{1}{A})u + C|p|$ for $x \in \Omega$,

$$0 \leq t < \infty,\ 0 \leq u \leq a, \text{ and } 0 \leq |p| \leq b,$$

where a and b are any fixed numbers > 0. Then, given $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ such that

if $\quad \sup_{x \in \Omega} u(x,0) \leq \delta$

and $\quad \sup_{x \in \partial\Omega} u(x,t) \leq \delta \qquad$ for $0 \leq t < +\infty$

then $\quad \sup_{x \in \bar{\Omega}} u(x,t) \leq \varepsilon \qquad$ for $0 \leq t < +\infty$

Proof: We may assume that $\varepsilon \leq \min(a, \frac{Ab}{B})$; if not, we may replace ε by something smaller. Take $\delta = \frac{\varepsilon}{A}$, and

$$v(x,t) = \frac{\varepsilon}{A}\ \phi(x), \text{ and apply Theorem 1 in } Q_\infty :$$

Since $\frac{\partial v}{\partial t} - L[v] \geq \frac{\varepsilon}{A} + \beta_o v + C|\nabla v|$

$$\geq \frac{1}{A}v + \beta_o v + C|\nabla v|$$

$$\geq F(x,t,v,\nabla v),$$

our proof is immediate, via Theorem 1.

COROLLARY. Suppose $u(x,t)$ satisfies (13) in Q_∞,

where $F(x,t,u,p) \leq \beta(x,t)u + C|p|$ for $x \in \bar{\Omega}$,

$0 \leq t < +\infty$, $0 \leq u \leq a$,

and $0 \leq |p| \leq b$;

furthermore, suppose $\sup\limits_{x \in \bar{\Omega}} \beta(x,t) \leq \beta_o + \frac{1}{A}$

and $\overline{\lim\limits_{t \to +\infty}} \left(\sup\limits_{x \in \bar{\Omega}} \beta(x,t) \right) < \beta_o$.

Then, $\exists\ \delta > 0$ such that if $\sup\limits_{x \in \bar{\Omega}} u(x,0) \leq \delta$,

$\sup\limits_{x \in \partial\Omega} u(x,t) \leq \delta$, for $0 \leq t < +\infty$,

and $\overline{\lim\limits_{t \to +\infty}} \left(\sup\limits_{x \in \partial\Omega} u(x,t) \right) \leq 0$

it follows that $\overline{\lim_{t \to +\infty}} \left(\sup_{x \in \bar{\Omega}} u(x,t) \right) \leq 0.$

Proof: Apply Theorem 6 first, choosing δ so small that $\sup_{x \in \bar{\Omega}} u(x,t) \leq \varepsilon$ for $0 \leq t < +\infty$, where ε is such that $A\varepsilon < a$ and $B\varepsilon < b$. Then apply Theorem 5, and the remark following it.

We conclude this section with a quantitative restatement of the corollary above, in which we predict how fast the solution decays, given more detailed information about F and the boundary values of u.

THEOREM 7. *Suppose* $u(x,t)$ *satisfies* (13) *in* Q_∞, *where*

$$F(x,t,u,p) \leq \alpha(x,t) + \beta(x,t)u + C|p| \quad \textit{for } x \in \bar{\Omega},$$

$$0 \leq t < +\infty, \ 0 \leq u \leq a, \textit{ and } 0 \leq |p| \leq b.$$

Let $N > 0$ *satisfy* $N \leq \frac{a}{A}$, $N \leq \frac{b}{B}$, *and suppose that* $\exists$ *two numbers* γ *and* δ *with* $0 \leq \gamma \leq \frac{1}{A}$ *and* $0 \leq \delta < +\infty$ such that

$$\sup_{x \in \bar{\Omega}} \alpha(x,t) \leq (1-\gamma A)Ne^{-\delta t},$$

$$\sup_{x \in \bar{\Omega}} \beta(x,t) + \delta \leq \gamma + \beta_o,$$

and $$\sup_{x \in \bar{\Omega}} u(x,t) \leq Ne^{-\delta t} \qquad \textit{for } 0 \leq t < +\infty.$$

Then, $$\sup_{x \in \bar{\Omega}} u(x,t) \le ANe^{-\delta t} \qquad \text{for } 0 \le t < +\infty .$$

Proof: Take $v(x,t) = Ne^{-\delta t}\,\phi(x)$; then $v \le AN \le a$, and $|\nabla v| \le BN \le b$. Thus, v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} - L[v] &\ge -\delta v + Ne^{-\delta t} + \beta_o v + C|\nabla v| \\ &\ge -\delta v + \alpha(x,t) + \gamma ANe^{-\delta t} + \beta_o v + C|\nabla v| \\ &\ge (\gamma+\beta_o-\delta)v + \alpha(x,t) + C|\nabla v| \\ &\ge F(x,t,v,\nabla v). \end{aligned}$$

We again complete our proof by applying Theorem 1.

The only thing even slightly remarkable about Theorems 6 and 7 is the local nature of the assumptions on F ; it should be emphasized, however, that our results merely assert that if a solution exists with small boundary values, its growth is determined. Nothing is said about whether such a solution exists, or how the gradient of a solution must behave.[1]

6. Finite Escape Times

We return here to a question raised in Section 4: When do solutions u(x,t) of (1), which satisfy given boundary conditions, become infinite as t --> T_o, where T_o is some finite number?

To answer this, we must be able to bound $\sup_{x \in \bar{\Omega}} u(x,t)$ from below; this we do in the following theorem.

THEOREM 8. We suppose that Ω is bounded, and that $u(x,t) \in C^{2,1}$ in Q_T, and satisfies there

$$\frac{\partial u}{\partial t} - L[u] \geq G(u,t) \tag{15}$$

where $L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j}\right)$ is a self-adjoint uniformly elliptic differential operator with smooth coefficients (in $C^3(\bar{\Omega})$, say) and $G(u,t)$ is a convex function of u for each fixed $t \geq 0$. Let $\phi(t)$ satisfy

$$\frac{d\phi}{dt} = G(\phi,t) - \lambda_1(\phi - k(t)) \qquad \text{for } 0 < t \leq T,$$

and

$$\phi(0) = \inf_{x \in \bar{\Omega}} u(x,0),$$

where $k(t) = \inf_{x \in \partial\Omega} u(x,t)$, and where λ_1 is the first eigenvalue for the problem

$$-L[\psi] = \lambda\psi \qquad \text{in } \Omega$$

with

$$\psi = 0 \qquad \text{on } \partial\Omega.$$

Then, we have $\sup_{x \in \bar{\Omega}} u(x,t) \geq \phi(t)$ for $0 \leq t \leq T$.

Proof: By Courant's Theorem, the eigenfunction $\psi(x)$ corresponding to λ_1, does not change sign in Ω. We may thus

take $\psi(x) \geq 0$ in Ω; furthermore we may assume that

$\int_\Omega \psi(x)dx = 1.$ We define

$$\hat{u}(t) = \int_\Omega u(x,t)\ \psi(x)dx \quad \left[\begin{array}{l} = (u(t),\psi), \text{ using the ordinary} \\ L^2\text{-scalar product notation} \end{array} \right]$$

We multiply both sides of (15) by $\psi(x)$, and integrate over Ω : We have $(\frac{\partial u}{\partial t}, \psi) = \frac{d\hat{u}}{dt}$, since u is continuously differentiable in t; furthermore,

$$-(L[u],\psi) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx - \int_{\partial\Omega} \frac{\partial \psi}{\partial \zeta}\, u\, d\sigma$$

by Stokes' Theorem, where $d\sigma$ is the element of n-1 dimensional surface area on $\partial\Omega$, and $\frac{\partial}{\partial \zeta}$ denotes differentiation with respect to the conormal direction field, given by

$$\zeta(x,t) = (\zeta_1,\dots,\zeta_n) \text{ where } \zeta_i(x,t) = \sum_{j=1}^{n} a_{ij}(x,t)\eta_j(x)$$

for $i = 1,\dots,n$. This is obviously an exterior direction field (see Section 1) since (a_{ij}) is positive definite. Since Stokes' Theorem gives also

$$-(L[\psi],u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx - \int \frac{\partial \psi}{\partial \zeta}\, u\, d\sigma$$

and since $\psi = 0$ on $\partial\Omega$, and (a_{ij}) is symmetric, we have

$$(-L[u],\psi) = -(L[\psi],u) + \int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, u\, d\sigma$$
$$= \lambda_1 \hat{u}(t) + \int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, u\, d\sigma .$$

Since $\int \psi\, dx$ is a positive measure of total mass = 1 on Ω, and since $G(u,t)$ is a convex function of u for each fixed t, Jensen's inequality gives us

$$\int_\Omega G(u(x,t),t)\ \psi(x)dx \geq G\left(\int_\Omega u(x,t)\ \psi(x)dx,t\right)$$
$$= G(\hat{u}(t),t).$$

Thus we obtain

$$\frac{du}{dt} \geq G(\hat{u}(t),t) - \lambda_1 \hat{u}(t) - \int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, u\, d\sigma .$$

Let us estimate the last term on the right hand side: Since ψ assumes its minimum value, zero, at every point of $\partial\Omega$, $\nabla\psi$ must be, at every such point, a vector pointing in the direction of the interior normal (but possibly null). Since ζ is an exterior direction field, we have $\frac{\partial\psi}{\partial\zeta} \leq 0$ for all $x \in \partial\Omega$; thus, $\int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, u\, d\sigma \leq k(t) \int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, d\sigma$. But, applying Stokes' Theorem again, with the functions ψ and 1, we obtain

$$\lambda_1 = \lambda_1 \int_\Omega \psi(x)dx = -(L[\psi],1) = -\int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta}\, d\sigma.$$

Therefore,

$$\int_{\partial\Omega} \frac{\partial\psi}{\partial\zeta} u \, d\sigma \le -\lambda_1 k(t); \quad \text{thus we have}$$

$$\hat{u}'(t) \ge G(\hat{u}(t),t) - \lambda_1(\hat{u}(t) - k(t)).$$

Since $u(0) = \phi(0)$, we may apply the special case of Theorem 1, mentioned in Section 1, to conclude that

$$\hat{u}(t) \ge \phi(t) \quad \text{for } 0 \le t \le T.$$

Since $\sup_{x \in \bar{\Omega}} u(x,t) \ge \hat{u}(t)$, our proof is complete.

In particular, if $\phi(t) \to +\infty$ as $t \to T_0$, so does $\sup_{x \in \bar{\Omega}} u(x,t)$. We illustrate this with the special case of the equation

$$\frac{\partial u}{\partial t} - L[u] = F(u) \tag{16}$$

where $F(u)$ is convex and positive for $u \ge u_0$, and $\int_{u_0}^{+\infty} \frac{du}{F(u)} < +\infty$. Suppose we try to find a solution $u(x,t)$

of (16) in Q_T, such that $m \leq u(x,0) \leq M$ for $x \in \bar{\Omega}$, and $k \leq u(x,t) \leq K$ for $x \in \partial\Omega$ and $0 \leq t \leq T$. How large can T be? We assume that m and k are large enough so that $m \geq m_o$ and $F(u) - \lambda_1(u-k) > 0$ for $u \geq m$. Then, Theorem 8 tells us that such a solution must become infinite as $t \to T_o$, where

$$T_o \leq \int_m^{+\infty} \frac{du}{F(u)-\lambda_1(u-k)} .$$

Moreover, the discussion in Section 4 assures us that

$$T_o \geq \int_{\max(M,K)}^{+\infty} \frac{du}{F(u)} .$$

Thus, we obtain estimates from above and below for the escape time for (15).

REFERENCES

[1] Courant, R. and Hilbert, D. - Methods of Mathematical Physics, Vol. I, Interscience, New York, 1953.

[2] Fillipov, A. - Conditions for the existence of a solution of a quasi-linear parabolic equation. Dok. Akad. Nauk. S.S.S.R., 141, 3, 1961, pp. 568-570.

[3] Friedman, A. - Remarks on the maximum principle for parabolic equations and its applications. Pac. J. of Math., 8, 2, 1958, pp. 201-211.

[4] Friedman, A. - Asymptotic behavior of solutions of parabolic equations of any order. Acta Math., 106, 1961, pp. 1-43.

[5] Hopf, E. - A remark on linear elliptic differential equations of second order. Proc. Am. Math. Soc., 3, 1952, pp. 791-793.

[6] Kruzhkov, S. and Oleinik, O. - Quasi-linear parabolic equations of second order in several space variables. Usp. Mat. Nauk., 16, 5, 1961, pp. 115-155.

[7] Magenes, E. and Stampacchia, G. - I problemi al contorno per le equazione differenziali di tipo ellitico. Ann. Sc. Norm. Sup. Pisa, 12, 3, 1958, pp. 247-357.

[8] Szarski, J. - Sur la limitation et l'unicité des solutions d'un systéme non-linéaire d'équations paraboliques aux dérivées partielles du second ordre. Ann. Polon.

Mat., 2, 2, 1955, pp. 237-249.

[9] Tykhonov, A. - Théorèmes d'unicité pour l'équation de la chaleur. Mat. Sb., 42, 1935, pp. 199-215.

[10] Walter, W. - Eindeutigkeitssätze für gewöhniliche, parabolische, und hyperbolische Differentialgleichungen. Math. Z., 74, 3, 1960, pp. 191-208.

[11] Westphal, H. - Zur abschätzung der Lösungen nichtlinearer parabolischer Differentialgleichungen. Math. Z., 51, 1949, pp. 690-695.

FOOTNOTE

1 It has been brought to our attention that similar results have been obtained, but only with assumptions on the global behavior of F as a function of p, by R. Narasimhan in his article, "On the asymptotic stability of solutions of parabolic differential equations", Journ. of Rat. Mech. and Anal., 3, 1954, pp. 303-313.

NYU
IMM-
305 c.1
Kaplan

On the growth of solutions of quasi-linear parabolic equations.

N. Y. U. Courant Institute of
Mathematical Sciences
4 Washington Place
New York 3, N. Y.